

## Environment-induced preresolution in the Mott-Hubbard model

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Via the hierarchy of correlations, we study the strongly interacting Fermi-Hubbard model in the Mott insulator state and couple it to a Markovian environment that constantly monitors the particle numbers  $\hat{n}_\mu^\uparrow$  and  $\hat{n}_\mu^\downarrow$  for each lattice site  $\mu$ . As expected, the environment induces an imaginary part  $\gamma$  (i.e., the decay rate) of the quasiparticle frequencies  $\omega_k \rightarrow \omega_k - i\gamma$ , and it tends to diminish the correlations between lattice sites. Surprisingly, the environment also steers the state of the system on intermediate timescales  $O(1/\gamma)$  to a preresolved state very similar to the prethermalized state after a quantum quench (i.e., suddenly switching on the hopping rate  $J$ ). Full relaxation or thermalization occurs via local on-site heating and takes much longer.

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### I. INTRODUCTION

Understanding the quantum dynamics of strongly interacting many-body systems is one of the major challenges of contemporary physics. Compared to weakly or noninteracting systems, strong interactions can induce new and fascinating phenomena. One example is the Mott insulator state: For a fermionic lattice with a half-filled band, one would expect conducting (i.e., metallic) behavior—but strong interactions can make this system insulating [1–3].

While the ground or thermal equilibrium state of strongly interacting systems may already display nontrivial properties, their nonequilibrium dynamics can pose even more difficult problems, which we are just beginning to understand. A conceptually clear and frequently studied example is a quantum quench, where one starts in the ground state of a given Hamiltonian and then suddenly (or nonadiabatically) changes one of the parameters of this Hamiltonian. After that, the initial state will no longer be the ground state in general, and the time dependence after such a global excitation has been studied in various works; see, e.g., [4–30].

One of the surprises and unexpected results of such nonequilibrium dynamics is the phenomenon of prethermalization: Even in systems that are expected to thermalize after a global excitation, this thermalization dynamics can occur in several stages with different timescales. Local observables that oscillate on short timescales (after the quench) approach a quasistatic value on intermediate timescales—which is, however, different from their thermal value. Full thermalization (if it occurs) requires much longer timescales. As an intuitive picture, prethermalization can be understood as dephasing of the quasiparticle excitations, while full thermalization requires the exchange of energy and momentum between the quasiparticles. How strongly interacting quantum many-body systems equilibrate is a very important and not fully solved question that has far-reaching consequences, ranging from solid-state devices such as the proposed Mott transistor

[31–33] or other switching processes to the observability of quark-gluon plasma.

So far, equilibration and thermalization dynamics of strongly interacting quantum many-body systems after quantum quenches and related questions have mostly been discussed in closed quantum systems undergoing a unitary evolution [4–24,34–52]. However, every real system is always coupled to an environment, which can also affect the equilibration and thermalization dynamics. To start filling this gap, we consider a prototypical example (1) for a strongly interacting quantum many-body system, and we study its nonequilibrium dynamics after coupling in an environment that is assumed to be Markovian.

### II. THE MODEL

The lattice system under consideration is described by the Fermi-Hubbard Hamiltonian ( $\hbar = 1$ )

$$\hat{H} = -\frac{1}{Z} \sum_{\mu,\nu,s} J_{\mu\nu} \hat{c}_{\mu,s}^\dagger \hat{c}_{\nu,s} + U \sum_{\mu} \hat{n}_\mu^\uparrow \hat{n}_\mu^\downarrow, \quad (1)$$

where  $\hat{c}_{\mu,s}^\dagger$  and  $\hat{c}_{\nu,s}$  are the fermionic creation and annihilation operators for the spin  $s \in \{\uparrow, \downarrow\}$  at the lattice sites  $\mu$  and  $\nu$ , respectively. The corresponding hopping rate is denoted by  $J_{\mu\nu}$ , where we have factored out the coordination number  $Z$ . The second term describes the on-site repulsion  $U$  with the particle number operators  $\hat{n}_\mu^\uparrow$  and  $\hat{n}_\mu^\downarrow$ . As possible experimental realizations, one could envision fermionic atoms in optical lattices [53–57] or electrons in solid-state settings [58,59].

The above Hamiltonian (1) generates the internal unitary evolution while the coupling to the Markovian environment is described in terms of a master equation with the Lindblad operators  $\hat{n}_{\mu,s}$  and the coupling strength  $\gamma$ ,

$$\partial_t \hat{\rho} = i[\hat{\rho}, \hat{H}] + \gamma \sum_{\mu,s} \left( \hat{n}_{\mu,s} \hat{\rho} \hat{n}_{\mu,s} - \frac{1}{2} \{ \hat{n}_{\mu,s}, \hat{\rho} \} \right), \quad (2)$$

where we have used  $\hat{n}_{\mu,s}^2 = \hat{n}_{\mu,s}$  for fermions. Thus, the environment permanently monitors (i.e., weakly measures) the number of particles  $\hat{n}_{\mu,s}$  per lattice site  $\mu$  for each spin species  $s$ . Such an environment could be represented by a bath of bosons that scatter off the fermions depending on their position. For example, for atoms in optical lattices they could be photons, and for electrons in solids they could be phonons.

From the above master equation (2), we may already infer properties of the steady state  $\hat{\rho}(t \rightarrow \infty) = \hat{\rho}^\infty$ , which should commute with all the (self-adjoint) Lindblad operators  $\hat{n}_{\mu,s}$  and with the Hamiltonian  $\hat{H}$ ; see, e.g., [60–62]. If we first focus on the Lindblad operators  $\hat{n}_{\mu,s}$ , the steady state  $\hat{\rho}_\mu^\infty$  of each lattice site  $\mu$  should be diagonal in the particle number basis, i.e., it could be an arbitrary incoherent sum of the zero-particle  $|0\rangle\langle 0|$ , one-particle  $|\uparrow\rangle\langle\uparrow|$  and  $|\downarrow\rangle\langle\downarrow|$ , as well as two-particle  $|\uparrow\downarrow\rangle\langle\uparrow\downarrow|$  states. However, due to the hopping terms  $\propto J$  in the Hamiltonian  $\hat{H}$ , the on-site particle numbers  $\hat{n}_{\mu,s}$  are no longer conserved separately, only the total numbers

$$\hat{N}_s = \sum_{\mu} \hat{n}_{\mu,s} \quad (3)$$

commute with  $\hat{H}$ . They are related to the total number  $\hat{N} = \hat{N}_\uparrow + \hat{N}_\downarrow$  and the  $z$  component of the total spin  $\hat{\Sigma}_z = (\hat{N}_\uparrow - \hat{N}_\downarrow)/2$ . Their (expectation) values are fixed by the initial conditions. Here, we consider the case of half-filling for each spin species (i.e., spin-unpolarized  $\langle \hat{\Sigma}_z \rangle = 0$ ), which also displays particle-hole symmetry. Hence, the steady state becomes unique,  $\hat{\rho}_\mu^\infty \propto \mathbf{1}$ , and corresponds to the infinite-temperature limit.

The above master equation (2) can also be written in terms of Liouville superoperators

$$\partial_t \hat{\rho} = \frac{1}{Z} \sum_{\mu,v} \mathfrak{L}_{\mu,v} \hat{\rho} + \sum_{\mu} \mathfrak{L}_{\mu} \hat{\rho}, \quad (4)$$

where  $\mathfrak{L}_{\mu,v}$  contains the hopping term  $\propto J_{\mu v}$  from (1) while the on-site interaction term  $\propto U$  from (1) as well as the environment contribution  $\propto \gamma$  from (2) are encoded in the local contribution  $\mathfrak{L}_{\mu}$ .

### III. HIERARCHY OF CORRELATIONS

Since the dynamics (2) can only be solved exactly for very small lattices (see Sec. VIII), we have to introduce a suitable approximation scheme. Here, we employ the hierarchy of correlations [21–24,63,64] and consider the reduced density matrices  $\hat{\rho}_\mu$  for one site and  $\hat{\rho}_{\mu v}$  for two sites, etc. After splitting off the correlations via  $\hat{\rho}_{\mu v}^{\text{corr}} = \hat{\rho}_{\mu v} - \hat{\rho}_\mu \hat{\rho}_v$  and so on, we obtain for the evolution of the on-site density matrix,

$$\begin{aligned} \partial_t \hat{\rho}_\mu &= \frac{1}{Z} \sum_v \text{Tr}_v \{ \mathfrak{L}_{\mu,v} \hat{\rho}_\mu \hat{\rho}_v + \mathfrak{L}_{\mu,v} \hat{\rho}_{\mu v}^{\text{corr}} \} + \mathfrak{L}_{\mu} \hat{\rho}_\mu \\ &= f_1(\hat{\rho}_v, \hat{\rho}_{\mu v}^{\text{corr}}). \end{aligned} \quad (5)$$

In analogy, the time evolution of the two-site correlations can be derived from (4) and also depends on the on-site density matrices as well as the three-site correlators,

$$\partial_t \hat{\rho}_{\mu v}^{\text{corr}} = f_2(\hat{\rho}_v, \hat{\rho}_{\mu v}^{\text{corr}}, \hat{\rho}_{\mu v \sigma}^{\text{corr}}). \quad (6)$$

To truncate this infinite set of recursive equations, we exploit the hierarchy of correlations in the formal limit of large coordination numbers  $Z \gg 1$ . With completely the same arguments as in [24], it can be shown that the two-site correlations are suppressed via  $\hat{\rho}_{\mu v}^{\text{corr}} = O(1/Z)$  in comparison to the on-site density matrix  $\hat{\rho}_\mu = O(Z^0)$  and the three-site correlators even stronger via  $\hat{\rho}_{\mu v \sigma}^{\text{corr}} = O(1/Z^2)$ , and so on. Note that the derivation in [24] works in completely the same way here because the environment acts locally, i.e., on each lattice site separately, and thus only changes the local Liouvillian  $\mathfrak{L}_\mu$  in (4).

This hierarchy of correlations facilitates the following iterative approximation scheme: To zeroth order in  $1/Z$ , we may approximate (5) via  $\partial_t \hat{\rho}_\mu \approx f_1(\hat{\rho}_v, 0)$ , which yields the mean-field solution  $\hat{\rho}_\mu^0$ . As the next step, we may insert this solution  $\hat{\rho}_\mu^0$  into (6) and obtain to first order in  $1/Z$  the following approximate set of linear and inhomogeneous equations for the correlations:

$$\partial_t \hat{\rho}_{\mu v}^{\text{corr}} \approx f_2(\hat{\rho}_v^0, \hat{\rho}_{\mu v}^{\text{corr}}, 0). \quad (7)$$

The solution of this set of equations describes the propagation (and damping) of the quasiparticles, and insertion back into (5) then yields their backreaction onto the mean field.

### IV. MEAN-FIELD ANSATZ

Let us study the propagation (and damping) of the quasiparticles according to (7) for a concrete example. For the mean-field solution  $\hat{\rho}_\mu^0$ , we assume a homogeneous and spin-symmetric (i.e., unpolarized) state, which can be described by the general ansatz

$$\hat{\rho}_\mu^0 = p_0 |0\rangle\langle 0| + p_1 (|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) + p_2 |\uparrow\downarrow\rangle\langle\uparrow\downarrow|, \quad (8)$$

with the probabilities for zero  $p_0$ , one  $p_1$ , and two particles  $p_2$  on the lattice site  $\mu$ . For the Fermi-Hubbard Hamiltonian (1) and the Lindblad operators  $\hat{n}_{\mu,s}$  in (2), this ansatz automatically satisfies the zeroth-order (mean-field) equation  $\partial_t \hat{\rho}_\mu^0 = f_1(\hat{\rho}_v^0, 0)$ .

Since we want to study the Mott insulator state [2,3], we assume half-filling  $\langle \hat{n}_\mu^\uparrow + \hat{n}_\mu^\downarrow \rangle = 2p_1 + 2p_2 = 1$ . Together with the normalization  $p_0 + 2p_1 + p_2 = 1$ , this fixes all probabilities except one, which can be parametrized by the double occupancy  $\mathfrak{D} = \langle \hat{n}_\mu^\uparrow \hat{n}_\mu^\downarrow \rangle = p_2$ . It vanishes in the Mott insulator state  $p_0 = p_2 = 0$ , but in the infinite-temperature limit  $p_0 = p_1 = p_2$ , it tends to  $1/4$ .

Note that, since the ansatz (8) obeys the zeroth-order (mean-field) equation  $\partial_t \hat{\rho}_\mu^0 = f_1(\hat{\rho}_v^0, 0)$ , the double occupancy  $\mathfrak{D}$  is constant to lowest order (in  $1/Z$ ). However, including the backreaction of the quasiparticles and their quantum or thermal fluctuations onto the mean field, it will change in general (see below).

### V. QUASIPARTICLES

Inserting the ansatz (8) into Eq. (7) for the correlations, we find the following set of relevant correlation functions (see also [21]):

$$\begin{aligned} f_{\mu v, s}^{00} &= \langle \hat{c}_{\mu, s}^\dagger (1 - \hat{n}_{\mu, \bar{s}}) \hat{c}_{v, s} (1 - \hat{n}_{v, \bar{s}}) \rangle, \\ f_{\mu v, s}^{01} &= \langle \hat{c}_{\mu, s}^\dagger (1 - \hat{n}_{\mu, \bar{s}}) \hat{c}_{v, s} \hat{n}_{v, \bar{s}} \rangle, \end{aligned}$$

$$\begin{aligned} f_{\mu\nu,s}^{10} &= \langle \hat{c}_{\mu,s}^\dagger \hat{n}_{\mu,\bar{s}} \hat{c}_{\nu,s} (1 - \hat{n}_{\nu,\bar{s}}) \rangle, \\ f_{\mu\nu,s}^{11} &= \langle \hat{c}_{\mu,s}^\dagger \hat{n}_{\mu,\bar{s}} \hat{c}_{\nu,s} \hat{n}_{\nu,\bar{s}} \rangle, \end{aligned} \quad (9)$$

with  $s \in \{\uparrow, \downarrow\}$  denoting the spin and  $\bar{s}$  the opposite spin. All other correlators vanish to first order (in  $1/Z$ ).

As we obtain the same dynamics for both spin species  $s$ , we omit the spin index  $s$  in the following. Assuming spatial homogeneity, we Fourier-transform the above correlation functions, and (7) becomes

$$\begin{aligned} (i\partial_t + i\gamma)f_k^{00} &= \frac{J_k}{2}(f_k^{10} - f_k^{01}) = -(i\partial_t + i\gamma)f_k^{11}, \\ (i\partial_t - U + i\gamma)f_k^{01} &= \frac{J_k}{2}(f_k^{11} - f_k^{00}) - \frac{J_k}{4}(1 - 4\mathfrak{D}) \\ &= -(i\partial_t + U + i\gamma)f_k^{10}. \end{aligned} \quad (10)$$

For time-independent parameters  $\gamma$ ,  $J_k$ ,  $U$ , and  $\mathfrak{D}$ , we may diagonalize the above linear system of equations and thereby obtain four eigenfrequencies. Two of them read

$$\omega_k^\pm = \pm\sqrt{U^2 + J_k^2} - i\gamma \quad (11)$$

while the other two are simply  $\omega_k^0 = -i\gamma$ . We see that all eigenfrequencies acquire the same imaginary part  $-i\gamma$ , which just corresponds to an exponential decay  $e^{-\gamma t}$ . This describes the damping of the quasiparticles induced by the coupling to the environment.

## VI. PRERELAXATION

Due to this exponential decay  $e^{-\gamma t}$ , the correlation functions approach the following asymptotic state (again assuming that  $\mathfrak{D}$  is constant):

$$\begin{aligned} f_{k,\text{asy}}^{00} &= \frac{J_k^2}{U^2 + J_k^2 + \gamma^2} \frac{1 - 4\mathfrak{D}}{4} = -f_{k,\text{asy}}^{11}, \\ f_{k,\text{asy}}^{01} &= \frac{J_k(U + i\gamma)}{U^2 + J_k^2 + \gamma^2} \frac{1 - 4\mathfrak{D}}{4} = (f_{k,\text{asy}}^{10})^*, \end{aligned} \quad (12)$$

which is independent of the initial state, i.e., the initial values  $f_k^{00}(t=0)$ ,  $f_k^{01}(t=0)$ ,  $f_k^{10}(t=0)$ , and  $f_k^{11}(t=0)$ . As one would expect, the correlations are suppressed for large  $\gamma$  and go to zero in the limit  $\gamma \rightarrow \infty$ .

However, this asymptotic state (12) differs from the final steady state (corresponding to infinite temperature and vanishing correlations) discussed in Sec. II. Thus, it describes a prerelaxed state and is not fully relaxed (thermalized) yet. Full relaxation or thermalization requires taking into account the backreaction (which changes  $\mathfrak{D}$ ); see Sec. VII below. It is also interesting to note that this asymptotic state (12) is different from the ground state, even for  $\gamma = 0$  and  $\mathfrak{D} = 0$  [21].

The latter can be determined from (10) via an adiabatic change of the hopping rate from  $J = 0$  to a finite value. For  $\gamma = 0$ , Eqs. (10) conserve the quantities

$$f_k^{00} + f_k^{11} = 0 \quad (13)$$

as well as

$$f_k^{01} f_k^{10} + (f_k^{11})^2 - \frac{f_k^{11}}{2} = 0. \quad (14)$$

This leads to the ground-state correlations

$$f_{k,\text{ground}}^{01} = f_{k,\text{ground}}^{10} = \frac{J_k}{4\sqrt{U^2 + J_k^2}} \quad (15)$$

and

$$f_{k,\text{ground}}^{00} = \frac{1}{4} \left( \frac{U}{\sqrt{U^2 + J_k^2}} - 1 \right) = -f_{k,\text{ground}}^{11}. \quad (16)$$

Already for small  $J_k$ , we observe a factor of 2 difference from the asymptotic state (12); see also [21,30].

### A. Universality

In fact, in the limit of small  $\gamma$ , the above asymptotic state (12) coincides with the prethermalized state after a quantum quench, where one starts in the ground state with  $J = 0$  and then suddenly switches on  $J$  to its final value; see, e.g., [21]. This coincidence seems to be a rather general property. To understand why, let us write the linear system of Eqs. (10) in matrix form,

$$\partial_t \mathbf{f}_k = \mathbf{M}_k \cdot \mathbf{f}_k - \gamma \mathbf{f}_k + \mathbf{s}_k, \quad (17)$$

with a time-independent matrix  $\mathbf{M}_k$  describing the Hamiltonian evolution, i.e., depending on  $J_k$  and  $U$ . Neglecting backreaction, i.e., assuming that the double occupancy  $\mathfrak{D} = \langle \hat{n}_\mu^\uparrow \hat{n}_\mu^\downarrow \rangle$  is time-independent, the source term  $\mathbf{s}_k$  is also constant. Then, due to the damping term  $\gamma$ , the correlations approach the asymptotic state

$$\mathbf{f}_k^{\text{asy}} = (\gamma \mathbf{1} - \mathbf{M}_k)^{-1} \cdot \mathbf{s}_k. \quad (18)$$

Now, the limit  $\gamma \rightarrow 0$  could be problematic if the source term  $\mathbf{s}_k$  had contributions in the kernel  $\ker(\mathbf{M}_k)$  of the matrix  $\mathbf{M}_k$ , i.e., the subspace of zero eigenvalue. In this case, the linear evolution according to (17) without environment  $\gamma = 0$  would imply linearly growing modes—which indicate an instability (e.g., if the mean-field ansatz  $\hat{\rho}_\mu^0$  does not describe a stable stationary state).

The kernel of  $\mathbf{M}_k$  is spanned by the vectors  $(-U, J_k, J_k, U)^T$  and  $(J_k, 0, 0, J_k)^T$ , which is orthogonal to the source term  $\mathbf{s}_k \propto (0, 1, -1, 0)^T$ . Thus we have  $\mathbf{s}_k \perp \ker(\mathbf{M}_k)$  and a linear growth of the modes cannot occur. Note that such a linear growth, when extrapolated to large times, would eventually lead to inconsistencies (such as negative probabilities). In the various scenarios investigated by us (see also [21]), we did not encounter this problem, i.e., we always found  $\mathbf{s}_k \perp \ker(\mathbf{M}_k)$ .

### B. Prethermalization

In the subspace orthogonal to the kernel  $\ker(\mathbf{M}_k)$  we may invert the matrix  $\mathbf{M}_k$ , and the limit  $\gamma \rightarrow 0$  of the asymptotic state (18) reads  $\mathbf{f}_k^{\text{asy}} = -\mathbf{M}_k^{-1} \cdot \mathbf{s}_k$ . Now let us compare this state to the prethermalized state after a quantum quench (without environment). If we start initially in the ground state for  $J = 0$ , we have vanishing correlations initially,  $\mathbf{f}_k(t=0) = 0$ . At time  $t = 0$ , we switch on the hopping rate  $J$ . The time evolution afterward can be obtained by solving (17) for

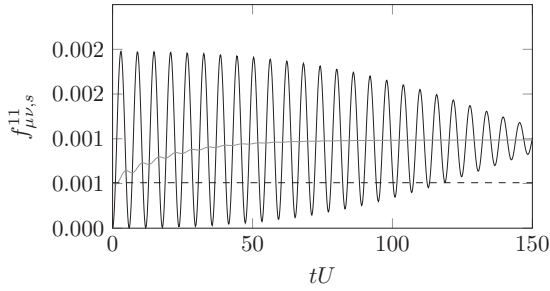


FIG. 1. Prethermalization and prere relaxation dynamics of the particle-particle correlator  $f_{\mu\nu,s}^{11}$  for next-to-nearest neighbors  $\mu$  and  $\nu$  in a cubic lattice without backreaction. The solid black curve corresponds to a quench without environment (prethermalization), where  $J$  is switched suddenly from zero to  $0.4U$ . The dashed line denotes the ground state at that value. The gray curve represents the dynamics for fixed  $J = 0.4U$  after starting in the ground state and switching on the coupling  $\gamma = 0.05U$  to the environment (pre-relaxation). Both curves approach approximately the same asymptotic state  $f_{\mu\nu,s}^{11,\text{asy}} \approx 10^{-3}$ .

$\gamma = 0$  and vanishing initial correlations, which yields

$$\mathbf{f}_k(t) = (\exp\{\mathbf{M}_k t\} - \mathbf{1}) \cdot \mathbf{M}_k^{-1} \cdot \mathbf{s}_k. \quad (19)$$

As a result, the Fourier modes  $\mathbf{f}_k(t)$  of the correlations oscillate with the (nonzero) eigenfrequencies of the matrix  $\mathbf{M}_k$ . The Fourier transformation back to position space then involves a sum over many Fourier modes with different oscillating phases, which gives the usual prethermalization dynamics as in Fig. 1. The long-time limit then corresponds to the time average  $\overline{\mathbf{f}_k}$  where the oscillating exponentials cancel  $\overline{\mathbf{f}_k} = -\mathbf{M}_k^{-1} \cdot \mathbf{s}_k$ . Hence, the coincidence of the prethermalized state (after a quench) and the  $\gamma \rightarrow 0$  limit of the asymptotic state with environment seems to be a general phenomenon—as long as arguments along the lines explained above apply.

Note that the simple matrix form (17) applies to cases in which all correlations are damped at the same rate  $\gamma$ . While this is true for the system under investigation [cf. (10)], one might have different damping rates  $\gamma_{1,2,\dots}$  for other scenarios. However, this just amounts to replacing  $\gamma\mathbf{1}$  by a different matrix (assumed to be positive-definite) while the rest of the arguments apply in the same way.

## VII. BACKREACTION

So far, we have neglected the backreaction of the quantum or thermal fluctuations of the quasiparticles onto the mean field and assumed that  $\hat{\rho}_\mu^0$  and thus the double occupancy  $\mathcal{D}$  are constant. To study this backreaction, we insert the (generally time-dependent) solutions  $f_k^{00}$ ,  $f_k^{01}$ ,  $f_k^{10}$ , and  $f_k^{11}$  back into (5), which gives

$$i\partial_t \mathcal{D} = \frac{2}{N} \sum_k J_k (f_k^{01} - f_k^{10}). \quad (20)$$

We see that even the asymptotic state (12) can induce a change of  $\mathcal{D}$  provided that  $\gamma \neq 0$ . For example, starting in the Mott insulator phase with zero or small  $\mathcal{D}$ , it would slowly grow due to local on-site heating induced by the coupling to the environment.

However, this growth rate is much slower than the damping of correlations and quasiparticles with their decay rate  $\gamma$ . From the above equation (20), we may estimate that this local on-site heating occurs on much longer timescales,

$$\tau_{\text{thermal}} \sim \frac{U^2}{J^2} \frac{Z}{\gamma} \gg \frac{1}{\gamma} = \tau_{\text{decay}}, \quad (21)$$

where the factor of  $Z$  stems from the Fourier transform (assuming an isotropic lattice).

For late times  $t \gg \tau_{\text{thermal}}$ , the double occupancy tends to  $1/4$  and thus all correlations between lattice sites vanish, as we may infer from (12). This final state  $\hat{\rho}_\mu \propto \mathbf{1}$  corresponds to the infinite-temperature limit already discussed in Sec. II, which is consistent with the fact that the considered Markovian environment acts as an infinite temperature heat bath; see Fig. 2.

## VIII. HUBBARD DIMER

To test the reliability of our approximation scheme, we also considered the exactly solvable case of the two-site Fermi-Hubbard model with one spin-up and one spin-down particle, i.e.,

$$\hat{H} = -J_d \sum_s (\hat{c}_{2,s}^\dagger \hat{c}_{1,s} + \hat{c}_{1,s}^\dagger \hat{c}_{2,s}) + U \sum_{\mu=1,2} \hat{n}_\mu^\uparrow \hat{n}_\mu^\downarrow. \quad (22)$$

To simplify the analysis further, we consider states that are fully symmetric with respect to a permutation of the lattice sites  $\mu = 1$  and  $\nu = 2$  and are invariant under spin-flips (i.e., unpolarized). Again, the on-site matrices  $\hat{\rho}_1 = \hat{\rho}_2$  can be fully parametrized by the double occupancy

$$\mathcal{D} = \langle \hat{n}_{1,s} \hat{n}_{1,\bar{s}} \rangle = \langle \hat{n}_{2,s} \hat{n}_{2,\bar{s}} \rangle \quad (23)$$

via the ansatz (8). Furthermore, as we only have one particle per spin species, the particle-particle and hole-hole correlators vanish, and only the symmetrized particle-hole correlator

$$\mathfrak{F} = \langle \hat{c}_{1,\bar{s}}^\dagger \hat{n}_{1,\bar{s}} \hat{c}_{2,s} (1 - \hat{n}_{2,\bar{s}}) \rangle + \langle \hat{c}_{2,\bar{s}}^\dagger \hat{n}_{2,\bar{s}} \hat{c}_{1,s} (1 - \hat{n}_{1,\bar{s}}) \rangle \quad (24)$$

survives. From the symmetries of the dimer state, it follows that  $\mathfrak{F} + \mathfrak{F}^* = 2\langle \hat{c}_{1,\bar{s}}^\dagger \hat{c}_{2,s} \rangle$ . Finally, the only remaining nonzero expectation values are two higher-order correlators,

$$\mathfrak{S} = \langle \hat{c}_{1,\bar{s}}^\dagger \hat{c}_{1,\bar{s}} \hat{c}_{2,\bar{s}}^\dagger \hat{c}_{2,s} \rangle, \quad (25)$$

$$\mathfrak{H} = \langle c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger c_{2\uparrow} c_{2\downarrow} \rangle + \langle c_{2\uparrow}^\dagger c_{2\downarrow}^\dagger c_{1\uparrow} c_{1\downarrow} \rangle. \quad (26)$$

With these definitions, the dimer dynamics is governed by the equations

$$i\partial_t \mathcal{D} = J_d (\mathfrak{F}^* - \mathfrak{F}), \quad (27)$$

$$(i\partial_t + U + i\gamma) \mathfrak{F} = -4J_d \mathcal{D} - 2J_d \mathfrak{S} + J_d \mathfrak{H} + J_d, \quad (28)$$

$$(i\partial_t - U + i\gamma) \mathfrak{F}^* = 4J_d \mathcal{D} + 2J_d \mathfrak{S} - J_d \mathfrak{H} - J_d, \quad (29)$$

$$(i\partial_t + 2i\gamma) \mathfrak{S} = J_d (\mathfrak{F}^* - \mathfrak{F}), \quad (30)$$

$$(i\partial_t + 4i\gamma) \mathfrak{H} = -2J_d (\mathfrak{F}^* - \mathfrak{F}). \quad (31)$$

In general there is no simple closed expression for the eigenvalues and eigenvectors of this linear set of equations.

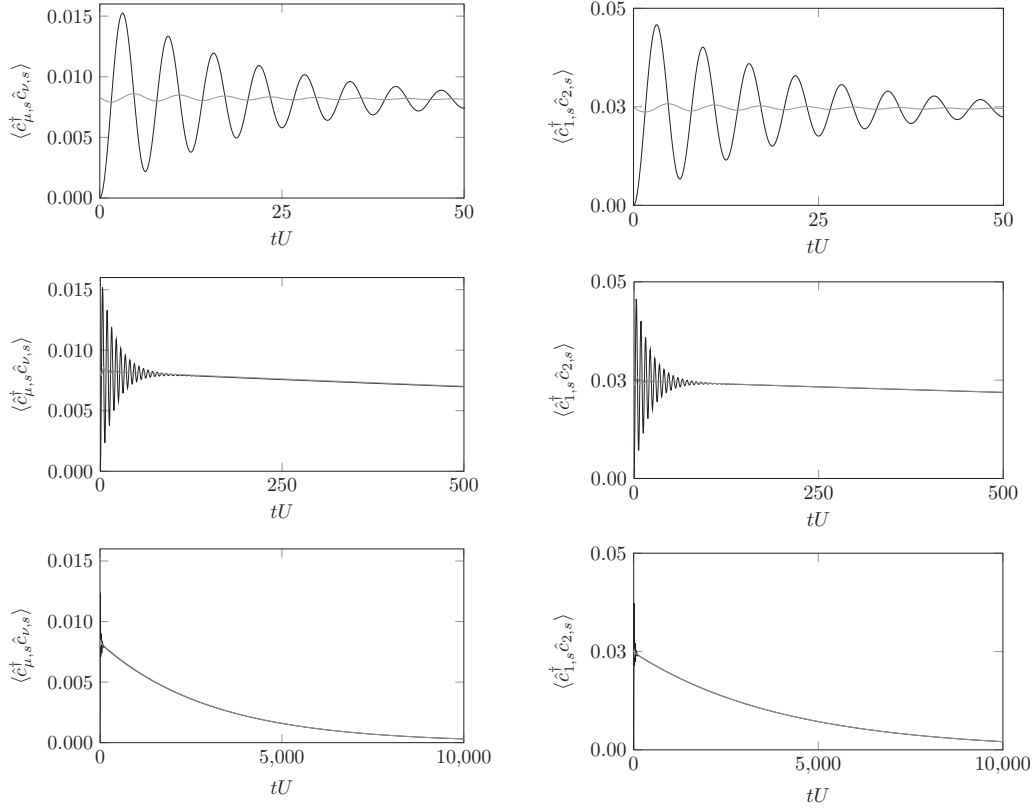


FIG. 2. Environment-induced relaxation dynamics of the correlation function  $\langle \hat{c}_{\mu}^\dagger \hat{c}_{\nu} \rangle$  for nearest neighbors including backreaction for the infinite cubic lattice (left column) and the Hubbard dimer (right column). In both cases, the coupling constant  $\gamma$  is chosen as  $\gamma = U/20$ . For the infinite cubic lattice (left column), we choose  $J = U/10$ , and for the Hubbard dimer (right column), we choose  $J_d = J/4 = U/40$  such that the  $\omega_{\pm}$  coincide with the eigenfrequencies (11) for  $\mathbf{k} = \mathbf{0}$ . As in Fig. 1, the black curves correspond to quantum quenches, where both  $J$  (or  $J_d$ ) and  $\gamma$  are suddenly switched on, while the gray curves start in the ground state at those values of  $J$  (or  $J_d$ ), and describe the evolution after  $\gamma$  is switched on. At short times (top row), we observe oscillations (with a frequency set by  $U$ ) that decay with the damping rate  $\gamma$ . Thus, at intermediate times (middle row), the state approaches a quasistationary state, which is quite independent of the initial conditions. At late times (bottom row), all correlations decay due to local on-site heating caused by the backreaction, and the system approaches the final infinite-temperature state.

Without dissipation, the two nonzero eigenvalues are  $\omega_{\pm} = \pm \sqrt{16J_d^2 + U^2}$ , in close analogy to (11).

To include the coupling to the environment, we consider the strongly interacting regime where  $J_d \ll U$ . Then, four eigenvalues acquire imaginary parts  $\sim \gamma$  and the correlation functions decay on a timescale of order  $1/\gamma$  to a prerelaxed state, which is given by

$$\mathfrak{D}_{\text{asy}} = \mathfrak{S}_{\text{asy}} = -\frac{\mathfrak{H}_{\text{asy}}}{2} = \frac{2J_d^2}{16J_d^2 + U^2}, \quad (32)$$

$$\tilde{\mathfrak{F}}_{\text{asy}} = \tilde{\mathfrak{F}}_{\text{asy}}^* = \frac{J_d U}{16J_d^2 + U^2}, \quad (33)$$

which is analogous to (12).

The remaining eigenvalue corresponds to the evolution of the double occupancy  $\mathfrak{D}$ , cf. (21), and is much smaller,

$$\lambda = -\frac{8i\gamma J_d^2}{U^2 + \gamma^2} + \mathcal{O}\left(\frac{J_d^3}{U^3}\right) \Rightarrow |\lambda| \ll \gamma. \quad (34)$$

The full relaxation or thermalization is governed by the above eigenvalue and thus occurs on much longer times. Again, due to local on-site heating, the system approaches an infinite-

temperature state  $\hat{\rho}_{12} \propto \mathbf{1}$ , which corresponds to  $\mathfrak{D} = 1/4$  and  $\tilde{\mathfrak{F}} = \mathfrak{S} = \mathfrak{H} = 0$ .

Note that, similar to our investigations based on the hierarchy of correlations, the asymptotic state of the Hubbard dimer (18) is different from its ground state,

$$\begin{aligned} \mathfrak{D}_{\text{ground}} = \mathfrak{S}_{\text{ground}} &= -\frac{\mathfrak{H}_{\text{ground}}}{2} \\ &= \frac{1}{8} \left( 1 - \frac{U}{\sqrt{16J_d^2 + U^2}} \right), \end{aligned} \quad (35)$$

$$\tilde{\mathfrak{F}}_{\text{ground}} = \tilde{\mathfrak{F}}_{\text{ground}}^* = \frac{J_d}{\sqrt{16J_d^2 + U^2}}. \quad (36)$$

## IX. CONCLUSIONS

As a prototypical example for a strongly interacting quantum many-body system, we consider the Fermi-Hubbard model (1) and couple it to a Markovian environment (2), which permanently performs weak measurements of the particle numbers  $\hat{n}_{\mu}^{\uparrow}$  and  $\hat{n}_{\mu}^{\downarrow}$  for each lattice site  $\mu$ . Via the hierarchy of correlations, we derive the evolution equations (10) for the

correlations, which are linear to first order (in  $1/Z$ ), as well as their backreaction (20) onto the mean field.

As expected, the coupling  $\gamma$  to the environment induces an imaginary part of the eigenfrequencies (11) leading to a decay of the quasiparticles and tends to suppress the correlations. Quite surprisingly, this damping mechanism also induces the phenomenon of prerelaxation quite analogous to prethermalization after a quantum quench. For small  $\gamma$ , the correlations even approach the same asymptotic state as after a quench. As our general arguments from (17) to (19) indicate, this seems to be a general phenomenon and shows that the environment-induced decoherence and damping of quasiparticles has a very similar effect to the dephasing of quasiparticles after a quench.

Taking the backreaction (20) into account, we find that the system eventually approaches a thermal state of infinite temperature. However, this on-site heating process is much slower and requires timescales (21) much longer than the intermediate timescale  $O(1/\gamma)$  of prerelaxation. Note that the emergence of different timescales in relaxation or thermalization processes has already been observed in other systems, e.g., for weak (or vanishing) interactions; see, e.g., [65–78].

Finally, we compared the results of our  $1/Z$  expansion with the exactly solvable case of two sites (Hubbard dimer) and found very similar results; see Fig. 2. We also considered the Mott-Néel state displaying antiferromagnetic spin ordering (see the Appendix) and found analogous behavior. In summary, the emergence of several stages characterized by different timescales in relaxation/thermalization processes seems to occur quite often, at least as long as some sort of linearized quasiparticle description of the form (17) is a good approximation.

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## APPENDIX

### Mott-Néel state

The propagation and damping of the quasiparticles is determined by the hierarchical equations up to first order in  $1/Z$ ,

$$\begin{aligned} i\partial_t f_{\mu\nu,s}^{ab} &= \frac{1}{Z} \sum_{\kappa,c} J_{\mu\kappa} \langle \hat{n}_{\mu,\bar{s}}^a \rangle f_{\kappa\nu,s}^{cb} - \frac{1}{Z} \sum_{\kappa,c} J_{\nu\kappa} \langle \hat{n}_{\nu,\bar{s}}^b \rangle f_{\mu\kappa,s}^{ca} \\ &+ \frac{J_{\mu\nu}}{Z} [\langle \hat{n}_{\mu,\bar{s}}^a \rangle \langle \hat{n}_{\nu,s}^1 \hat{n}_{\nu,\bar{s}}^b \rangle - \langle \hat{n}_{\nu,\bar{s}}^b \rangle \langle \hat{n}_{\mu,s}^1 \hat{n}_{\mu,\bar{s}}^a \rangle] \\ &- \left[ \frac{(-1)^b - (-1)^a}{2} U + i\gamma \right] f_{\mu\nu,s}^{ab} + O(1/Z^2), \end{aligned} \quad (\text{A1})$$

where we used the shorthand notation  $\hat{n}_{\mu,s}^1 = \hat{n}_{\mu,s}$  and  $\hat{n}_{\mu,s}^0 = 1 - \hat{n}_{\mu,s}$ . The correlations induce a backreaction onto the mean field, which changes the double occupancy according to

$$i\partial_t \mathcal{D} = \frac{1}{Z} \sum_{\kappa,s} J_{\mu\kappa} (f_{\kappa\mu,s}^{01} - f_{\mu\kappa,s}^{10}) + O(1/Z^2). \quad (\text{A2})$$

After a Fourier transformation of (A1) for a spatially homogeneous system at half-filling, one obtains the set of equations (10).

Since the hierarchical set of equations (A1) is derived in real space, we are not restricted to spatially homogeneous systems. For example, the fermionic Hubbard system in a cubic or hypercubic lattice prefers to be in a staggered Mott-Néel state with sublattices *A* and *B* if the temperature is sufficiently low. Assuming to lowest order a perfect staggering,  $\langle \hat{n}_s^A \rangle = \langle \hat{n}_s^B \rangle = 1$ , the Fourier components of the correlation functions satisfy the equations

$$(i\partial_t + i\gamma) f_{k,s}^{00,AA} = J_k (f_{k,s}^{10,BA} - f_{k,s}^{01,AB}), \quad (\text{A3})$$

$$(i\partial_t + i\gamma) f_{k,s}^{11,BB} = J_k (f_{k,s}^{01,AB} - f_{k,s}^{10,BA}), \quad (\text{A4})$$

$$(i\partial_t - U + i\gamma) f_{k,s}^{01,AB} = J_k (f_{k,s}^{11,BB} - f_{k,s}^{00,AA}) - J_k, \quad (\text{A5})$$

$$(i\partial_t + U + i\gamma) f_{k,s}^{10,BA} = J_k (f_{k,s}^{00,AA} - f_{k,s}^{11,BB}) + J_k. \quad (\text{A6})$$

The system of equations can be diagonalized and we obtain, similar to Eq. (11), two  $\mathbf{k}$ -dependent eigenfrequencies

$$\omega_k^\pm = \pm \sqrt{U^2 + 4J_k^2} - i\gamma \quad (\text{A7})$$

and two  $\mathbf{k}$ -independent eigenfrequencies, which read  $\omega_k^0 = -i\gamma$ . After a quantum quench, the correlation functions approach the prethermalized state

$$f_{k,s}^{11,BB} = -f_{k,s}^{00,AA} = \frac{2J_k^2}{\gamma^2 + 4J_k^2 + U^2}, \quad (\text{A8})$$

$$f_{k,s}^{01,AB} = (f_{k,s}^{10,BA})^* = \frac{J_k(U + i\gamma)}{\gamma^2 + 4J_k^2 + U^2}, \quad (\text{A9})$$

which is independent of the initial correlations  $f_k(t=0)$ . Again, the asymptotic state is different from the ground state, even for  $\gamma = 0$ , where we have

$$f_{k,s}^{11,BB} = -f_{k,s}^{00,AA} = \frac{1}{2} \left( 1 - \frac{U}{\sqrt{4J_k^2 + U^2}} \right), \quad (\text{A10})$$

$$f_{k,s}^{01,AB} = f_{k,s}^{10,BA} = \frac{J_k}{\sqrt{4J_k^2 + U^2}}. \quad (\text{A11})$$

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